

# Determination of two-body potentials from $n$ -body spectra

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## Abstract

We show how the two-body potential may be uniquely determined from  $n$ -body spectra in cases where the hypercentral approximation is valid. We illustrate this by considering an harmonic oscillator potential which has been altered by changing the energy or normalisation constant of the ground state of the  $n$ -body system and finding how this modifies the two-body potential. It is shown that with increasing number of particles the spectrum must be known more precisely to obtain the two-body potential to the same degree of accuracy.

# 1 Introduction

The standard inverse scattering theory is formulated for two-body systems [1]. Its aim is the determination of the interaction from the spectral information pertaining to bound and scattering states of the two particles. In some cases there is no or insufficient spectral information available on the two-body system, while there is access to the spectral data of a corresponding many-body system. An example of this is the baryon spectrum whereas no two-quark system is known to exist. Thus it is equally interesting to ask whether it is possible to extract the two-body potential from the bound and scattering states of a many-body system. To our knowledge this general question has never been considered. There is a schematic treatment of the three-body problem in one space dimension by Zakhariev [3], but this is unsuitable for practical applications. With respect to applications, only recently [4] has a feasible method for the inversion of baryon spectra been proposed and have calculations of the associated quark-quark potentials been reported.

In the following, spectral information given in terms of many-body bound states is considered. For many-body bound states there are approximations which bear a formal resemblance to a two-body Schrödinger equation. This property suggests the possibility of applying standard two-body inversion techniques within these approaches. Examples of these include the Hartree-Fock method in which the interaction of the bodies is described by a mean field, and the hypercentral approximation (HCA) in which only the hyperradial part of the interaction in the hyperspherical harmonic expansion method [2] (HHEM) is retained. It is on the latter scheme, described in more detail in the following section, that we will focus in this article. The hypercentral potential used in this method is determined uniquely by the underlying two-body interaction. The main progress achieved in this study is an inversion of this relation for a general  $n$ -body system consisting of  $n$  identical particles. This inversion procedure is formulated for arbitrary  $n$  and can be applied as long as the HCA gives a proper description of the many-body system. In principle the method is an extension of the inversion procedure of baryon spectra, investigated in previous papers [4, 5]. In this extended scheme we show how the two-body interaction may be obtained in general from  $n$ -body spectral information, and in particular use the method to obtain the two-body interaction from three-, four-, five- and six-body spectral data.

We illustrate this scheme by applying our inversion method to spectra which coincide with that of the harmonic oscillator in both energy levels, and normalisation constants of the bound states in hyperspherical space, bar the ground state. This ground state is either shifted in energy relative to the harmonic oscillator ground state or has a different bound state normalisation constant than that of the harmonic oscillator ground state. Primarily we focus on the changes of the two-body potential with increasing number of particles  $n$ . Our example clearly demonstrates that the accuracy of the determination of the two-body potential from the  $n$ -body spectrum deteriorates with increasing  $n$ , if the accuracy to which the bound states are given remains fixed. It implies a rapid loss of information on the underlying two-particle potential with increasing size of the bound system of particles, despite the corresponding increase in the number of states present in their spectrum. This can also be turned around and viewed as an indication that the  $n$ -body spectrum becomes less sensitive to the details of the two-body interaction with increasing number of particles.

The details of the inversion in the HCA are described in the following section, and

the illustration and the results are to be found in section 3. We conclude in section 4, discussing what may be learnt from the example we have used.

## 2 The hypercentral approximation and inversion formalism

We begin this section with a description of the HCA and in particular show the relation of the hypercentral potential to Wigner-type two-body interactions. We then show how this relation may be inverted, thereby facilitating an inversion of  $n$ -body spectral data to obtain the underlying two-body interaction.

### 2.1 The hypercentral approximation

An  $n$ -body state is described after removing the centre-of-mass motion by  $3(n-1)$  internal co-ordinates. The corresponding  $n$ -particle Schrödinger equation can generally only be solved using various approximations. A very effective procedure is the expansion of the wave functions and potentials in terms of hyperspherical harmonics. This HHEM is particularly useful for the evaluation of many-body bound states of both bosons and fermions.

Specifically for fairly soft two-body interactions, where correlations between the particles are negligible, the lowest order of the HHEM, the so-called hypercentral approximation (HCA), provides an excellent description of the many-body bound states. It is exact in the case of the  $n$ -body harmonic oscillator, where it is well known that the  $n$ -body bound states may be obtained as linear combinations of single particle oscillator eigenfunctions. A prominent example of the applicability of the HCA is the calculation of baryon spectra [6] in the framework of the non-relativistic quark model, in which the forces operating between the constituent quarks are soft and confining. Even for a Coulomb interaction or certain nucleon-nucleon interactions, the HCA alone may account for up to 90 % of the binding energy of an  $n$ -body bound state of the system (see [2] and references therein).

In the HCA the  $n$ -body Schrödinger equation reduces to an effective Schrödinger equation in the hyperradius  $\rho$

$$\rho^2 = \frac{2}{n} \sum_{i < j}^n (\mathbf{r}_i - \mathbf{r}_j)^2, \quad (1)$$

where  $\mathbf{r}_i$  denotes the co-ordinate of the  $i$ -th particle. The hypercentral potential  $V_{hc}^n(\rho)$  which accounts for the interactions of the  $n$  bodies is the rotationally invariant part of the sum of two-body central potentials  $U(|\mathbf{r}_i - \mathbf{r}_j|)$  acting between each pair of particles  $i$  and  $j$  in the  $3(n-1)$ -dimensional space. It is determined by the two-body potential  $U$  through the linear relation [2, 7]

$$\begin{aligned} V_{hc}^n(\rho) &= \sum_{j=0}^{2l_m} V_j^n(\rho) \\ &= \sum_{j=0}^{2l_m} \frac{a_j}{h} \int_{-1}^{+1} dz (1-z)^{\frac{D-5}{2}+L_m-j} (1+z)^{j+\frac{1}{2}} U\left(\rho \sqrt{\frac{1+z}{2}}\right), \end{aligned} \quad (2)$$

where  $D$  has been used to denote  $3(n-1)$ , and  $l_m$  is the maximum value of the orbital angular momentum in the outermost shell consistent with a hyperspherical harmonic of

degree  $L_m$ . By allowing the minimal degree  $L_m$  of hyperspherical harmonics present in the expansion of the  $n$ -body wavefunction to be greater than zero, the Pauli Principle may be implemented for bound states of fermions. The normalisation constant  $h$  is defined by

$$h = \sum_{j=0}^{2L_m} a_j \int_{-1}^{+1} dz (1-z)^{\frac{D-5}{2}+L_m-j} (1+z)^{j+\frac{1}{2}}. \quad (3)$$

The  $a_j$  are constants determined by the shell structure of the bound state. If  $L_m = 0$ , as is the case for all  $n$ -boson bound states, the relation simplifies to

$$V_{hc}^n(\rho) = \frac{\Gamma(\frac{D}{2})}{\sqrt{\pi} 2^{D/2-2} \Gamma(\frac{D-3}{2})} \int_{-1}^{+1} dz (1-z)^{\frac{D-5}{2}} (1+z)^{\frac{1}{2}} U(\rho \sqrt{\frac{1+z}{2}}). \quad (4)$$

Thus  $V_{hc}^n(\rho)$  represents some weighted average of the two-body potential over the range  $(0, \rho)$ , reflecting the range of the possible separations of pairs of particles in the  $n$ -body state of hyperradius  $\rho$ . The total  $n$ -body interaction due to the  $\frac{n(n-1)}{2}$  pairs of particles interacting via the two-body interaction  $U$  is then accounted for by  $\frac{n(n-1)}{2} V_{hc}^n(\rho)$ . For  $n$  particles of equal mass  $m$ , a bound state with energy  $E$  is determined in the HCA by the solution of

$$\left\{ \frac{\hbar^2}{m} \left[ -\frac{d^2}{d\rho^2} + \frac{\mathcal{L}(\mathcal{L}+1)}{\rho^2} \right] + \frac{n(n-1)}{2} V_{hc}^n(\rho) - E \right\} u(\rho) = 0 \quad (5)$$

where the so-called grand orbital momentum  $\mathcal{L}$  depends on the orbital angular momentum  $l$  and the dimension of the space,

$$\mathcal{L} = l + L_m + \frac{D-3}{2}. \quad (6)$$

For all two-body potentials which are less singular than  $r^{-3}$  at the origin, the hypercentral potential is well-defined. This implies that the hypercentral potential is also no more singular than this, as the leading order behaviour of the hypercentral potential is that of the two-body potential near the origin. This condition does not in reality restrict the use of the HCA, as an attractive potential must be less singular than  $r^{-2}$  if the spectrum is to be bounded from below, while the presence of a repulsive singularity in the potential would signal the possible inappropriateness of the HCA, as correlations between the particles could be important.

In physical systems where the HCA provides a good description of the spectrum,  $n$ -body spectral data could be used via standard inversion techniques to obtain the effective interaction

$$\frac{\mathcal{L}(\mathcal{L}+1)}{\rho^2} + \frac{m}{\hbar^2} \frac{n(n-1)}{2} V_{hc}^n(\rho) \quad (7)$$

appearing in equation (5). The corresponding two-body potential may then be deduced from the hypercentral potential via a further inversion step.

In the following we present such a procedure. As the expression is more complex if  $L_m > 0$ , we first derive the relation for an  $n$ -boson hypercentral potential, and then use these results to generalise to the  $n$ -fermion case.

## 2.2 The inversion of the $n$ -boson hypercentral potential

Here we consider the inversion of the hypercentral potential in the simplest case, where  $L_m = 0$ . Using the new variables

$$x = \rho^2 \quad \text{and} \quad y = \rho^2 \left( \frac{1+z}{2} \right), \quad (8)$$

we rewrite the relation (4) in a form which we can readily invert to obtain the two-body potential  $U$  in terms of the hypercentral potential  $V_{hc}^n$ ,

$$\frac{\sqrt{\pi} \Gamma\left(\frac{D-3}{2}\right)}{2\Gamma\left(\frac{D}{2}\right)} x^{\frac{D-2}{2}} V_{hc}^n(\sqrt{x}) = \int_0^x dy \sqrt{y} (x-y)^{\frac{D-5}{2}} U(\sqrt{y}). \quad (9)$$

We consider two distinct cases, depending on whether the system consists of an odd or even number of particles.

1. If  $n$  is even, we may simply differentiate the above expression to obtain the two-body potential in terms of the hypercentral potential. Setting  $D - 5 = 2k$ , where  $k = 2, 5, 8, \dots$ , we find

$$U(\sqrt{y}) = \frac{\sqrt{\pi}}{2} \frac{1}{\Gamma\left(k + \frac{5}{2}\right)} \frac{1}{\sqrt{y}} \frac{d^{k+1}}{dy^{k+1}} \left[ y^{k+\frac{3}{2}} V_{hc}^n(\sqrt{y}) \right]. \quad (10)$$

2. If  $n$  is odd, we write  $D - 5 = 2k + 1$ , where  $k = 0, 3, 6, \dots$ , and on differentiating expression (9) we obtain

$$\frac{d^{k+1}}{dx^{k+1}} \left[ x^{k+2} V_{hc}^n(\sqrt{x}) \right] = \frac{2}{\pi} \frac{\Gamma(k+3)}{\pi} \int_0^x dy \frac{\sqrt{y} U(\sqrt{y})}{\sqrt{x-y}}. \quad (11)$$

This is in the form of the Abel integral equation [8],

$$g(s) = \int_0^s dt \frac{f(t)}{\sqrt{s-t}} \quad (12)$$

which may be uniquely inverted to obtain

$$f(t) = \frac{1}{\pi} \int_0^t ds \frac{dg}{ds} \frac{1}{\sqrt{t-s}} + \frac{1}{\pi} \frac{g(0)}{\sqrt{t}}. \quad (13)$$

The Abel transform is a special case of the Riemann-Liouville fractional integral [9]. Thus we can invert (11) to obtain

$$U(\sqrt{y}) = \frac{1}{2} \frac{1}{\Gamma(k+3)} \frac{1}{\sqrt{y}} \left[ \int_0^y \frac{d^{k+2}}{dx^{k+2}} \left[ x^{k+2} V_{hc}^n(\sqrt{x}) \right] \frac{dx}{\sqrt{y-x}} + \frac{1}{\sqrt{y}} \lim_{x \rightarrow 0} \frac{d^{k+1}}{dx^{k+1}} \left[ x^{k+2} V_{hc}^n(\sqrt{x}) \right] \right]. \quad (14)$$

The derivation of the two-body potential from the three-body hypercentral potential presented previously [4, 5] is an example of the above transformation with  $k = 0$ .

While these relations above show that it is in principle possible to extract the two-body interaction from a hypercentral potential describing a system of arbitrary  $n$ , it is clear from the form of these relations that in practice the larger the number of particles, the more difficult it is to perform this inversion numerically as higher order derivatives enter the relation. We will demonstrate this quantitatively in the following section, following the generalisation to the  $n$ -fermion case.

## 2.3 The inversion of the $n$ -fermion hypercentral potential

Using the variables  $x$  and  $y$  as defined in(8), we may re-express each  $V_j^n(\rho)$  appearing in (2) as

$$V_j^n(\sqrt{x}) = \frac{a_j}{h} \left(\frac{2}{x}\right)^{\frac{D-5}{2}+L_m+\frac{3}{2}} \int_0^x dy (x-y)^{\frac{D-5}{2}+L_m-j} y^{j+\frac{1}{2}} U(\sqrt{y}). \quad (15)$$

Then  $V_j^n$  may be found in terms of the two-body potential as described in the preceeding section.

If there are an even number of fermions, setting  $D - 5 = 2k$ , where  $k = 2, 5, 8, \dots$ , we find

$$A_j y^{j+\frac{1}{2}} U(\sqrt{y}) = \frac{d^{k+L_m-j+1}}{dy^{k+L_m-j+1}} \left[ y^{k+L_m+\frac{3}{2}} V_j^n(\sqrt{y}) \right], \quad j = 0, 1, \dots, 2l_m, \quad (16)$$

where we have defined the constant  $A_j$  as

$$A_j = 2^{k+L_m+\frac{3}{2}} (k + L_m - j)! \frac{a_j}{h}. \quad (17)$$

Taking the  $j^{th}$  derivative with respect to  $y$  of each of the above expressions and then summing over  $j$  we find the relationship from which  $U$  may be determined from a known hypercentral potential:

$$\sum_{j=0}^{2l_m} A_j \frac{d^j}{dy^j} \left[ y^{j+\frac{1}{2}} U(\sqrt{y}) \right] = \frac{d^{k+L_m+1}}{dy^{k+L_m+1}} \left[ y^{k+L_m+\frac{3}{2}} V_{hc}^n(\sqrt{y}) \right]. \quad (18)$$

For an odd number of fermions, writing  $D - 5 = 2k + 1$ , where  $k = 0, 3, 6, \dots$ , we have

$$\frac{d^{k+L_m-j+1}}{dx^{k+L_m-j+1}} \left[ x^{k+L_m+2} V_j^n(\sqrt{x}) \right] = B_j \int_0^x dy \frac{y^{j+\frac{1}{2}} U(\sqrt{y})}{\sqrt{y-x}} \quad (19)$$

where the constant  $B_j$  is defined as

$$B_j = \frac{\Gamma\left(k + \frac{3}{2} + L_m - j\right) 2^{k+\frac{3}{2}+L_m}}{\sqrt{\pi}} \frac{a_j}{h}. \quad (20)$$

If  $V_{hc}^n(x)$  is no more singular than  $x^{-2}$ , then use of the Abel transform once again leads to the expressions

$$\begin{aligned} B_0 y^{\frac{1}{2}} U(\sqrt{y}) &= \frac{1}{\pi} \int_0^y \frac{d^{k+L_m+2}}{dy^{k+L_m+2}} \left[ x^{k+L_m+2} V_0^n(\sqrt{x}) \right] \frac{dx}{\sqrt{y-x}} \\ &+ \frac{1}{\pi \sqrt{y}} \lim_{x \rightarrow 0} \frac{d^{k+L_m+1}}{dx^{k+L_m+1}} \left[ x^{k+L_m+2} V_0^n(\sqrt{x}) \right] \end{aligned} \quad (21)$$

and

$$B_j y^{j+\frac{1}{2}} U(\sqrt{y}) = \frac{1}{\pi} \int_0^y \frac{d^{k+L_m-j+2}}{dx^{k+L_m-j+2}} \left[ x^{k+L_m+2} V_j^n(\sqrt{x}) \right] \frac{dx}{\sqrt{y-x}}, \quad j = 1, 2, \dots, 2l_m. \quad (22)$$

Taking derivatives with respect to  $y$  and performing an integration by parts we find

$$\begin{aligned} B_j \frac{d^j}{dy^j} \left[ y^{j+\frac{1}{2}} U(\sqrt{y}) \right] &= \frac{1}{\pi} \int_0^y \frac{d^{k+L_m+2}}{dx^{k+L_m+2}} \left[ x^{k+L_m+2} V_j^n(\sqrt{x}) \right] \frac{dx}{\sqrt{y-x}} \\ &+ \frac{1}{\pi} \frac{1}{\sqrt{y-x}} \frac{d^{k+L_m+1}}{dx^{k+L_m+1}} \left[ x^{k+L_m+2} V_j^n(\sqrt{x}) \right] \Big|_{x=0}, \quad j = 1, 2, \dots, 2l_m \end{aligned} \quad (23)$$

Summing these expressions over  $j$  we obtain a differential equation

$$\begin{aligned} \sum_{j=0}^{2l_m} B_j \frac{d^j}{dy^j} \left[ y^{j+\frac{1}{2}} U(\sqrt{y}) \right] &= \frac{1}{\pi} \int_0^y \frac{dx^{k+L_m+2}}{dx^{k+L_m+2}} \left[ x^{k+L_m+2} V_{hc}^n(\sqrt{x}) \right] \frac{dx}{\sqrt{y}} \\ &+ \frac{1}{\pi \sqrt{y}} \frac{dx^{k+L_m+1}}{dx^{k+L_m+1}} \left[ x^{k+L_m+2} V_{hc}^n(\sqrt{x}) \right] \Big|_{x=0} \end{aligned} \quad (24)$$

as for the even number of fermions.

Thus in order to obtain the underlying two-body potential from an  $n$ -fermion hypercentral potential, a differential equation of order  $2l_m$  must be solved. Note that if  $L_m = 0$ , as we have not considered non-Wigner type interactions, the  $n$ -fermion result coincides with that of the  $n$ -boson result. We may have  $L_m = 0$  if we are dealing with fermions with some additional degree of freedom such as isospin or colour. Examples of such systems include bound states of three quarks or three or four nucleons.

### 3 The example of an-almost-harmonic oscillator

The two step procedure, introduced in the previous section, allows the determination of the two-body potential from the knowledge of the spectral information of the  $n$ -body system. In principle this spectral information for bound states includes both the energy levels and the normalisation constants, which represent additional information on the wave functions. Both quantities are required to fully determine the potential. Usually the energy levels, which are directly related to the mass spectrum of the bound system, can be measured with high accuracy. The situation is less satisfactory for the normalisation constants, which are not directly accessible to experiment and can only be deduced from transition observables. Consequently there is a considerable uncertainty in the normalisation constants.

It is precisely this situation which we want to study in a first schematic example. Given this new  $n$ -body inversion procedure, we would like to have an idea of what may be learnt about two-body interactions from  $n$ -body spectral data. To this end, we perform an inversion of  $n$ -body spectral information to obtain the two-body interaction in the simplest case, where  $L_m = 0$ . As a basic model then, we consider a bound system of  $n$  bosons of equal mass  $m$  interacting via two-body harmonic oscillator potentials  $\frac{r^2}{b^4}$ . This system has only a discrete spectrum and is exactly described by the HCA. The corresponding hypercentral potential is also a harmonic oscillator potential,  $V_{hc}^n = (n-1)^{-1}U$ , and the  $n$ -body bound state is obtained by solving a Schrödinger-type equation in the hyperradius  $\rho$

$$\left\{ -\frac{d^2}{d\rho^2} + \frac{\mathcal{L}(\mathcal{L}+1)}{\rho^2} + \frac{\rho^2}{b_n^4} - \epsilon \right\} u(\rho) = 0. \quad (25)$$

The energy of this state is  $\epsilon \frac{m}{\hbar^2}$  and the potential term is simply given by

$$V = \frac{m}{\hbar^2} \frac{n(n-1)}{2} V_{hc}^n = \frac{\rho^2}{b_n^4}. \quad (26)$$

The solution is of the form

$$u(\rho) = \left( \frac{\rho}{b_n} \right)^{\mathcal{L}+1} \exp \left( \frac{-\rho^2}{2b_n^2} \right) {}_1F_1(\alpha, \beta, \left( \frac{\rho}{b_n} \right)^2), \quad (27)$$

where  ${}_1F_1$  is the confluent hypergeometric function of the arguments

$$\alpha = (2\mathcal{L} + 3 - \epsilon b_n^2)/4 \quad (28)$$

and

$$\beta = \mathcal{L} + \frac{3}{2}. \quad (29)$$

If  $\alpha = -n$ , where  $n$  is an integer, one obtains a bound state and (27) becomes

$$u(\rho) = N \left( \frac{\rho}{b_n} \right)^{\mathcal{L}+1} \exp \left( \frac{-\rho^2}{2b_n^2} \right) L_n^{\beta-1} \left( \frac{\rho}{b_n} \right)^2, \quad (30)$$

where  $N$  is a constant normalising the wave function, and  $L_n^{\beta-1}$  denotes a Laguerre polynomial. In particular, if  $\alpha = 0$ , corresponding to  $\epsilon = \epsilon_0 = (2\mathcal{L} + 3)/b_n^2$ , we obtain the ground state  $u_0$

$$u_0 = \left[ \frac{b_n}{2} \Gamma \left( \mathcal{L} + \frac{3}{2} \right) \right]^{-\frac{1}{2}} \left( \frac{\rho}{b_n} \right)^{\mathcal{L}+1} \exp \left( \frac{-\rho^2}{2b_n^2} \right). \quad (31)$$

It is not our intention to reconstruct the two-body harmonic oscillator potential from the corresponding  $n$ -body bound-state spectrum because this can be done analytically. Rather, we want to simulate the realistic situation taking into account the uncertainties in the normalisation constants or energies. Therefore we study the variations of the two-body potential due to slight changes in the  $n$ -body spectrum. In order to demonstrate the essential points in a simple way we restrict ourselves to small modifications of the  $n$ -body ground states. We alter either the ground state normalisation constant leaving the energy unchanged, or the ground state energy, keeping the same ground state normalisation constant. All other remaining spectral data of the  $n$ -body system are unchanged from those of the  $n$ -body harmonic oscillator.

Because of our well defined reference potential it is not necessary to invert this modified  $n$ -body spectrum via a standard inversion procedure in order to obtain the corresponding hypercentral potential. Techniques of supersymmetric quantum mechanics [10] offer a more elegant way of achieving the same result in our case. This method is directly related to the standard inversion procedure of Gel'fand- Levitan [11] and has the additional advantage that we can really take into account the whole  $n$ -body spectrum. The latter point will be very important for the interpretation of our final result because we can be sure that all variations found in the two-body potential are generated by the modification of the  $n$ -body ground-state normalisation constant. There will be no artifacts due to the neglect of  $n$ -body states, which is unavoidable in a direct application of standard inversion methods.

We alter the  $n$ -body spectrum via supersymmetric transformations in two steps, first, by removing the harmonic oscillator ground state, and then by inserting a new ground state into the  $n$ -body spectrum. To remove the ground state from the  $n$ -body spectrum, we perform the transformation

$$\tilde{V} = V - 2 \frac{d^2}{d\rho^2} \ln u_0. \quad (32)$$

This results in the new potential

$$\tilde{V} = \frac{\rho^2}{b_n^4} + \frac{2}{b_n^2} + \frac{2(\mathcal{L} + 1)}{\rho^2}. \quad (33)$$

The solutions  $\tilde{u}$  associated with this new potential may be obtained from equation (27) with the replacements  $\mathcal{L} \rightarrow \mathcal{L} + 1$  and  $\epsilon \rightarrow \epsilon - \frac{2}{b_n^2}$ , thus

$$\tilde{u}(\rho) = \left(\frac{\rho}{b_n}\right)^{\mathcal{L}+2} \exp\left(-\frac{\rho^2}{2b_n^2}\right) {}_1F_1(\alpha + 1, \beta + 1, \left(\frac{\rho}{b_n}\right)^2). \quad (34)$$

Adding a new state to the spectrum is accomplished by the transformation

$$\tilde{\tilde{V}} = \tilde{V} - 2\frac{d^2}{d\rho^2} \ln \tilde{u} (1 + \lambda \int_\rho^\infty \tilde{u}^{-2}(z) dz) \quad (35)$$

where  $\lambda$  is a positive constant which controls the bound state normalisation constant of the new  $n$ -body ground state.

We need to check that the behaviour of  $\tilde{\tilde{V}}$  near the origin does not violate the conditions necessary for construction of the corresponding two-body potential. Expanding  $\tilde{\tilde{V}} - \tilde{V}$  to leading- and next-to-leading order for small  $\rho$ , we find that the change in the potential is

$$-2(\ln \tilde{u})'' - 2(\ln(1 + \lambda \int_\rho^\infty \tilde{u}^{-2}(z) dz))'' = -\frac{2\mathcal{L} + 2}{\rho^2} - \frac{4}{b_n^2} \left(\frac{\alpha + 1}{\beta + 1} - \frac{1}{2}\right) \left(1 - 2\frac{2\mathcal{L} + 3}{2\mathcal{L} + 1}\right) + O(\rho^3), \quad (36)$$

where  $'$  denotes differentiation with respect to  $\rho$ . The  $\rho^{-2}$  term can immediately be seen to cancel the singularity which arises from the removal of the original ground state.

If we insert a state at the same energy as the ground state of the harmonic oscillator, this corresponds to choosing  $\tilde{u}$  with  $\alpha = 0$ , and the constant in the above expression becomes

$$-\frac{4}{b_n^2} \left(\frac{1}{\beta + 1} - \frac{1}{2}\right) \left(1 - 2\frac{2\mathcal{L} + 3}{2\mathcal{L} + 1}\right) = -\frac{4}{b_n^2} \frac{-\mathcal{L} - \frac{1}{2} - 2\mathcal{L} - 5}{2\mathcal{L} + 5} \frac{2\mathcal{L} - 5}{2\mathcal{L} + 1} = -\frac{2}{b_n^2} \quad (37)$$

which leaves the potential unchanged at the origin. A change in the energy of the ground state will however produce a shift in the potential at the origin.

If one merely wishes to change the normalisation constant of the ground state one can choose  $\tilde{u} = u_0^{-1}$  [10] and the transformations (32) and (35) combine to

$$\tilde{\tilde{V}} = V - 2(\ln(1 + \kappa \int_0^\rho u_0^2(z) dz))'', \quad (38)$$

where use has been made of the fact that  $u_0$  is the normalized ground state, and the constant  $\kappa$  is related to the  $\lambda$  of the transformation shown in equation (35) by

$$\kappa = \frac{-\lambda}{\lambda + 1}. \quad (39)$$

Here the relation between the wave function of the new ground state and that of the original ground state near the origin is determined by the choice of  $\kappa$  [10]

$$\lim_{\rho \rightarrow 0} \tilde{u}(\rho) = \sqrt{1 + \kappa} \lim_{\rho \rightarrow 0} u_0(\rho). \quad (40)$$

In our first numerical study we focus on the sensitivity of the hypercentral and corresponding two-body potential to the  $n$ -body spectral data at a fixed particle number  $n$  and orbital angular momentum quantum number  $l = 0$ . Using the four-body case as an

example, we vary either the size of the normalised four-body ground state wavefunction at the origin, or shift the energy by  $\pm 5\%$  and  $\pm 10\%$ . The changes induced in the potential are depicted in Figure 1. While the change in the potential gets larger as the change in the energy or normalisation constant increases, it is obvious that even for these relatively small changes the potential does not respond linearly.

We turn our attention to the relative changes in potential that occur if we make the same change to the spectra of different number of bodies. In Figures 2(a) and (b) we show the change in  $V_{hc}^n$  induced by a reduction of the bound state normalisation constant by 5 % and the lowering of the ground state energy by 5 % without change in the normalisation constant respectively for three, four, five and six bodies. The corresponding changes induced in the two-body potentials appear in Figures 3(a) and (b). For comparison, we include the results in the case of a bound state of two bodies. In Figures 3(c) and (d) we show the full new two-body potentials, to illustrate that the change to the oscillator potential is not so drastic as to invalidate the use of the HCA.

Turning first to the changes in  $V_{hc}^n$  we see that the greater the number of bodies, the further from the origin the maximum change in the potential occurs. This is to be expected from the form of the HCA: with an increasing number of bodies, the effective centrifugal barrier as determined by  $\mathcal{L}$  increases, and sensitivity to the potential in this region is reduced. The change in  $V_{hc}^n$  decreases as  $n$  increases, which provides hope for the possibility of obtaining a reliable hypercentral potential from the inversion of  $n$ -body spectral data. However, this decrease is not transmitted to the underlying two-body interaction, as becomes clear on scrutiny of Figure 2. Here the change induced in the two-body potential is seen to increase rapidly with increasing particle number, even though the change in the corresponding hypercentral potential is decreasing.

Also worth noting, is that the number of oscillations in the two-body potential increases with increasing number of bodies. From the form of the transformation (10), it is clear that  $p$  maxima and minima in the hypercentral potential translate into  $p + k + 1$  extrema in the two-body potential. This has obvious implications for a potential obtained through inversion, which may have spurious numerically induced oscillations due to the neglect of part of the original  $n$ -body spectrum. The results clearly indicate the limitations on any  $n$ -body inversion.

## 4 Conclusions

We have derived a relation for the exact inversion of the underlying two-body potential from a given  $n$ -body hypercentral potential. In principle this implies that in physical systems which are adequately described by the hypercentral approximation, we have obtained an exact solution of the  $n$ -body inverse spectral problem in terms of two-body forces.

We have studied the possibility of extracting information on the two-body potential from  $n$ -body spectral data in the schematic example of an-almost-harmonic oscillator. It turns out that with increasing number of particles the  $n$ -body state becomes less sensitive to details of the underlying two-body interaction. In other words a small uncertainty in the  $n$ -body spectral data translates into a dramatic change in the two-body potential. This is especially true of any uncertainty in the bound state normalisation constant.

The harmonic oscillator is a fairly good approximation to some realistic systems, e.g.

basic features of light nuclei can be obtained within oscillator models. We are therefore justified in transposing our results to realistic systems. In particular they imply that the accuracy of the bound state measurements for larger systems must increase considerably if we want to get comparable information on the two-body potential. This situation also corresponds to an intuitive understanding of the increasing difficulty of extracting the contribution to the interaction of a pair of particles, from the total interaction of the  $\frac{n(n-1)}{2}$  pairs, which has been accounted for only in an average way. From this point of view it appears to be a further indication of the fundamental conjecture that entropy must increase in any many-to-one mapping which is nowhere formally proven.

Our numerical examples indicate that the inversion of  $n$ -body spectra of large systems to determine the underlying two-body potential is subject to increasing loss of information. However this does not mean that important information cannot be obtained from large systems, in addition to that which may be extracted from smaller systems.

The method remains at present restricted to cases where the hypercentral approximation is valid, that is, for weakly correlated systems of particles. An extension of spectral inversion for  $n$ -body spectra interacting via forces which induce strong correlations between particles still remains a formidable problem for the future.

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# References

- [1] Z.S. Agranovitsch and V.A. Marchenko, “*The Inverse Problem of Scattering Theory*”, Gordon, New York (1963);  
I.M. Gel’fand and B.M. Levitan, *Izv. Akad. Nank. SSR, ser. Mat.* 109 (1951).
- [2] M. Fabre de la Ripelle and J. Navarro, *Ann. Phys.* **123**, 185 (1979).
- [3] B.N.Zakhariev, *Few-Body Systems* **4**, 25 (1988).
- [4] H.Leeb, H.Fiedeldey, E.J.O. Gavin, S.A. Sofianos and R. Lipperheide, *Few-Body Systems* **12**, 55 (1992).
- [5] E.J.O. Gavin, H. Fiedeldey, H. Leeb and S.A. Sofianos, to appear in *Int. J. Mod. Phys. A*.
- [6] J.M. Richard and P. Taxil, *Ann. Phys. (N.Y.)* **150**, 267 (1983); *Phys. Lett.* **128B**, 453 (1983).
- [7] M. Fabre de la Ripelle, H. Fiedeldey and M. Wiechers, *Ann. Phys.* **138**, 275 (1982);  
R.M. Adam, S.A. Sofianos, H. Fiedeldey and M. Fabre de la Ripelle, *J. Phys. G* **18**, 1365 (1992).
- [8] R. Courant and D. Hilbert, “*Methods of Mathematical Physics, Vol. I*”, Interscience, New York (1962).
- [9] A. Erdélyi (ed.), “*Tables of integral transforms, Vol. II*”, McGraw-Hill Book Company, Inc., New York (1954).
- [10] C.V. Sukumar, *J. Phys. A Math. Gen.* **18**, 2937 (1985).
- [11] P.B. Abraham and H.E. Moses, *Phys. Rev. A* **22**, 1333 (1980).

## Figure Captions

1. Figure 1: The change produced in the four-body hypercentral and corresponding two-body potential if (a) the bound state normalisation constant is altered, and (b) the energy of the ground state is altered by  $\pm 5\%$  (indicated by dashed and dotted lines respectively) and  $\pm 10\%$  (the dash-dotted and dash-double dotted lines).
2. Figure 2: The changes to the hypercentral potential where (a) the bound state normalisation constant of the ground state is decreased by 5 % and (b) the ground state energy level is lowered by 5 % for three, four, five and six bodies denoted by the dashed, dotted, dash-dotted and dash-double dotted lines respectively.
3. Figure 3: The changes to the two-body potentials (a) and (b), corresponding to those in Figure 2 (a) and (b), as well as the total new two-body potential (c) and (d). The solid line indicates the (change in) potential for the two-body case.

Figure 1(a)

Figure 1(b)

Figure 2(a)

Figure 2(b)